Ergodic Theory and Measured Group Theory
Lecture 21

Bernoulli shifts.

Example. Let $\mathbb{Z} \curvearrowright^{s} (k^{\mathbb{Z}}, \nu^{\mathbb{Z}})$ be the Bernoulli shift action and ν a prob. meas. on $k := \{0, 1, \dots, k-1\}$. Let $\mathcal{P}$ be the base partition of $k^{\mathbb{Z}}$, i.e. $\mathcal{P} = \{P_0, P_1, \dots, P_{k-1}\}$, where $P_i := \{x \in k^{\mathbb{Z}} : x(0_{\mathbb{Z}}) = i\}$. This is a generating partition, so by the Kolmogorov–Sinai theorem, $h(S) = h(\mathcal{P}, S) = -\sum_{i<k} \nu(i) \log \nu(i) =: h(\nu)$.

$\forall \gamma, \delta \in \mathbb{Z}, \ \gamma \cdot x(\delta) := x(\delta + \gamma)$.

Cor (Kolmogorov). The Bernoulli actions $\alpha := (2^{\mathbb{Z}}, \{\frac{1}{2}, \frac{1}{2}\}^{\mathbb{Z}})$ and $\beta := (3^{\mathbb{Z}}, \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}^{\mathbb{Z}})$ are not isomorphic. (Indeed, $h(\alpha) = \log 2$ and $h(\beta) = \log 3$.)

Remark. For a sofic group Γ, Bowen showed that $\alpha_\Gamma := (2^{\Gamma}, \{\frac{1}{2}, \frac{1}{2}\}^{\Gamma})$ and $\beta_\Gamma := (3^{\Gamma}, \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}^{\Gamma})$ are not isomorphic by defining a notion of entropy for sofic groups. Note that free groups are sofic, so we have a notion of entropy for nonamenable groups, wow!
But for arbitrary cbl groups, we don't have it;

in particular, it's still open whether α and β are isomorphic for an arbitrary Γ.

It is clear that entropy is an isomorphism invariant, i.e. isomorphic actions have equal entropy. Ornstein show the converse to this for Bernoulli shift (of $\mathbb{Z}$).

Theorem (Ornstein 1980). Let $\mathbb{Z} \curvearrowright^{\alpha} (X^{\mathbb{Z}}, \mu^{\mathbb{Z}})$ and $\mathbb{Z} \curvearrowright^{\beta} (Y^{\mathbb{Z}}, \nu^{\mathbb{Z}})$ by shift, where (X, μ) and (Y, ν) are st. prob. spaces (e.g. $X := k$ and $\mu := \{\frac{1}{k}, \frac{1}{k}, \dots, \frac{1}{k}\}$ and $Y := [0,1)$, $\nu := \lambda$).

$\alpha \cong \beta \iff h(\alpha) = h(\beta)$.

Call $T \in \mathrm{Aut}(\mu)$ **Bernoulli** if $\mathbb{Z} \curvearrowright^{T} (X, \mu)$ is isomorphic to a Bernoulli shift, i.e. shift action $\mathbb{Z} \curvearrowright (Y^{\mathbb{Z}}, \nu^{\mathbb{Z}})$.

Let $\mathrm{BerAut}(\mu)$ be the set of Bernoulli automorphism $\in \mathrm{Aut}(\mu)$

Lemma (Ornstein). $\mathrm{BerAut}(\mu)$ is a Borel subset of $\mathrm{Aut}(\mu)$. In particular, it is a standard Borel (Any Borel subset of a Polish is st. Borel, by DST.)

Thus, Ornstein's theorem says that the isomorphism relation on $\mathrm{BerAut}(\mu)$ is smooth (= concretely classifiable). This is a great example of a smooth eq. rel. I know only two other ones...

Detour: other smooth eq. rel.

(a) Finitely generated abelian groups. Indeed, the space of such groups is std Borel (it's a Borel subset of $2^{(\mathbb{N}^3)}$) and by the classification theorem, each such group $\Gamma \cong \mathbb{Z}^n \oplus (\text{finite abelian group})$ and the map $\Gamma \mapsto (n, \text{finite abelian gp}) \in \mathbb{N}^2$ a Borel map, witnessing the smoothness of the isomorphism relation.

(b) Similarity of $n \times n$ complex matrices $M_n(\mathbb{C})$. Recall $A \sim B :\Leftrightarrow \exists Q \in GL_n(\mathbb{C})$ $QAQ^{-1} = B \Leftrightarrow A$ and B are conjugate $\Leftrightarrow A \; E_{GL_n(\mathbb{C})} \; B$ for the conjugation action $GL_n(\mathbb{C})$.
Recall from linear algebra that $A \sim B \Leftrightarrow J(A) = J(B)$, where $J(A)$ is the Jordan canonical form. The map $A \mapsto J(A)$ is Borel, so it's a Borel selector for $\sim$

on $M_n(\mathbb{C})$, i.e. it selects one winner matrix from each conjugacy class, in particular witnessing the smoothness of $\sim$. In particular, $\sim$ is Borel (unlike what its definition suggests).

Back to ergodic theory. We saw that in general the isomorphism of pmp actions of $\mathbb{Z}$ is difficult to understand (only successful on small subsets of $\mathrm{Aut}(\mu)$, e.g. $\mathrm{BerAut}(\mu)$). For more complicated groups Γ, it's even more complex. But ergodic theory is being developed for other groups..., including entropy theory.

Measured group theory

As we saw, ergodic theory focuses on studying, for a fixed ctbl group Γ, its pmp or more generally quasi-pmp/nonsingular actions, where the action is what's of interest. Measured gp theory focuses on studying ctbl groups by looking at its pmp (or quasi-pmp) actions, where the group is what's of interest,

> "Groups, as [people], will be known by their actions."
>
> —Guillermo Moreno

More specifically, we (meas. gp. thy) study the "average" behavior of groups in the following sense: one would like to equip a given ctbl gp Γ with a translation invariant prob. measure, but this is only possible when Γ is finite. When Γ is infinite, what we do instead is look at a free pmp action of Γ on a st. prob space (X,μ), so each orbit is a "copy" of Γ by freeness and any two points in the same orbit (i.e. in the same copy of Γ) have equal "weight" by pmp-ness.

Def. Let $\Gamma \curvearrowright (X,\mu)$ be an action of a ctbl gp Γ on a st. prob. space. Call this action **free** if $\forall \gamma \in \Gamma$ nonidentity, $\gamma x \neq x$ $\forall x \in X$. This ensures that $\gamma \mapsto \gamma \cdot x$ is a bijection with the orbit $[x]_\Gamma$. Call this action **measure preserving (pmp)** if each $\gamma \in \Gamma$ acts as a pmp automorphism of (X,μ), i.e. $\mu(\gamma \cdot A) = \mu(A)$ $\forall A \subseteq X$.

In fact, we will be concerned with the orbit eq. rel. of a free pmp action of Γ, as opposed to the particular action itself. In other words, the relevant equivalence relation is not the isomorphism of two actions, but their orbit equivalence.

Def. Let E, F be CBERs on st. meas. spaces (X,μ) & (Y,ν). Call E & F measure isomorphic if there is meas. isom $\pi : (X,\mu) \to (Y,\nu)$ (i.e. $\pi_* \mu = \nu$ & π is almost injective) s.t. π(E-class) = F-class, more precisely, $\forall x_1, x_2 \in X$, $x_1 \, E \, x_2 \iff \pi(x_1) \, F \, \pi(x_2)$.

Def. Let Γ, Δ be ctbl groups. Their pmp actions $\Gamma \curvearrowright^{\alpha} (X,\mu)$ & $\Delta \curvearrowright^{\beta} (Y,\nu)$ are call orbit equivalent if their orbit eq. rels E_α & E_β are measure-isomorphic. This is much coarser than isomorphism & it also makes sense when $\Gamma \neq \Delta$. We denote this by $\alpha \; \mathrm{OE} \; \beta$.